On the solution of the double Wiener-Hopf equation involved in the quarter plane problem

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V. Daniele[1] G. Lombardi[2]

Abstract – **This paper provides a semi analytical procedure to factorize the two dimensional kernel involved in the quarter plane diffraction problem. The proposed method is based on the reduction of the factorization problem to the solution of a Fredholm integral equation of second kind. The solution of the Fredholm integral equation appears cumbersome since it involves two folded integrals. In order to reduce the number of the numerical unknowns a suitable representation of the W-H unknowns is proposed.**

1 INTRODUCTION

The Wiener-Hopf formulation of the diffraction problem by a quarter plane has been accomplished in two well-known works by Radlow [1, 2]. The fundamental task of these papers was to get the factorization of a function of two complex variables. Radlow provided a solution of this difficult problem. However, even though several authors gave credit to this solution [3], now it definitively appears the Radlow factorization is wrong [4].
Recently Daniele and Lombardi proposed a method to obtain approximate factorizations of function of one complex variable [5]. This method is based to the reduction of the factorization problem to the solution of a Fredholm integral equation of second kind and it has been proved that it is very efficient to factorize the kernels involved in important diffraction canonical problems [6, 7].
The aim of this paper is to extend the above Fredholm formulation to the factorization of the two dimensional kernel involved in the quarter plane diffraction problem. Even though this reduction was successfully accomplished, the problem of the numerical quadrature of two folded integral remains cumbersome. In order to reduce the numerical complexity a suitable representation of the W-H unknowns is proposed.

2 THE WIENER-HOPF EQUATION

Let us consider a soft quarter of plane in plane $z = 0$ defined by: $z = 0$, $x \geq 0, y \geq 0$. The source of the problem is constituted by the incident plane wave defined in(1).

$$\psi^i(x, y, z) = e^{-j\eta_o x - j\xi_o x - j\alpha_o z} \quad (1)$$

k is the propagation constant in the free space and θ_o and φ_o are the azimuthal and zenithal angles of the incident plane wave; and $\eta_o = k \sin\theta_o \cos\varphi_o$, $\xi_o = k \sin\theta_o \sin\varphi_o$, $\alpha_o = \sqrt{k^2 - \eta_o^2 - \xi_o^2} = k\cos\theta_o$.

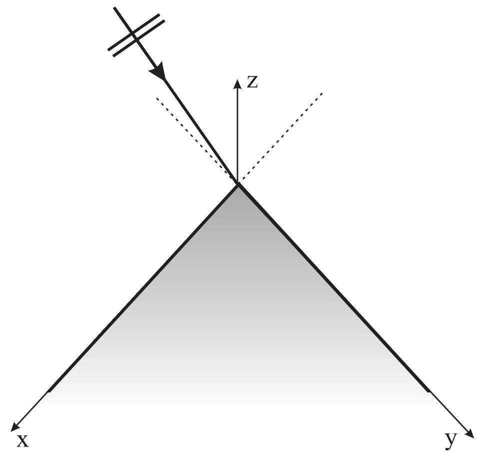

Figure 1: Geometry of the problem.

By indicating with $\psi(x, y, z)$ the total field, we must solve the equation:

$$\nabla^2\psi + k^2\psi = 0 \quad (2)$$

with the following boundary condition: $\psi(x, y, 0) = 0$ $x \geq 0$, $y \geq 0$.

The scattered field $\psi^s(x, y, z)$ satisfies the equation:

$$\nabla^2\psi^s + k^2\psi^s = j(x, y)\delta(z) \quad (3)$$

where $j(x, y)$ is the equivalent source induced on the quarter plane.
To get the mathematical meaning of $j(x, y)$ we apply the operator $\int_{0_-}^{0_+} []dz$ to the above equation. It yields:

[1] Dipartimento di Elettronica, Politecnico di Torino, C.so Duca degli Abruzzi 24, 10129 Torino, Italy, e-mail: vito.daniele@polito.it, fax: +39 011 5644099.
[2] Dipartimento di Elettronica, Politecnico di Torino, C.so Duca degli Abruzzi 24, 10129 Torino, Italy, e-mail: guido.lombardi@polito.it, fax: +39 011 5644099.

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$$\left.\frac{\partial\psi(x,y,z)}{\partial z}\right|_{z=0_+}-\left.\frac{\partial\psi(x,y,z)}{\partial z}\right|_{z=0_-}=j(x,y) \quad (4)$$

Taking into account the relationship:

$$(\nabla^2+k^2)\frac{e^{-jk\sqrt{(x-x')^2+(y-y')^2+z^2}}}{\sqrt{(x-x')^2+(y-y')^2+z^2}}= \quad (5)$$
$$=-4\pi\delta(x-x')\delta(y-y')\delta(z)$$

the following Green representation holds:

$$\psi(x,y,z)=\psi^i(x,y,z)+$$
$$-\frac{1}{4\pi}\int_0^\infty\int_0^\infty\frac{e^{-jk\sqrt{(x-x')^2+(y-y')^2+z^2}}}{\sqrt{(x-x')^2+(y-y')^2+z^2}}j(x',y')dx'dy' \quad (6)$$

Assuming the points $z=0, x\geq 0, y\geq 0$ where $\psi(x,y,z)=0$, we obtain the two-dimensional W-H equation:

$$\int_0^\infty\int_0^\infty g(x-x',y-y')j(x',y')dx'dy'=\psi^i(x,y,0) \quad (7)$$
$$x\geq 0, y\geq 0$$

where: $g(x,y)=\frac{1}{4\pi}\frac{e^{-jk\sqrt{x^2+y^2}}}{\sqrt{x^2+y^2}}$

The double Fourier Transform of $g(x,y)$ is evaluating using polar coordinate system and it is given by:

$$G(\eta,\xi)=\int_{-\pi}^{\pi}\int_0^\infty\frac{e^{-jk\rho}}{4\pi\rho}e^{j\eta\rho\cos\varphi}e^{j\xi\rho\sin\varphi}\rho d\rho d\varphi=\frac{-j}{4\pi}\int_{-\pi}^{\pi}\frac{1}{k-\eta\cos\varphi-\xi\sin\varphi}d\varphi$$
$$G(\eta,\xi)=-j\frac{1}{2}\frac{1}{\sqrt{k^2-\eta^2-\xi^2}}$$

It yields the following equation in the spectral domain:

$$G(\eta,\xi)F_+(\eta,\xi)=F_+^i(\eta,\xi)+F_-(\eta,\xi) \quad (8)$$

where:

$$F_+^i(\eta,\xi)=J_{++}(\eta,\xi)=\int_{-\infty}^\infty\int_{-\infty}^\infty j(x,y)e^{j\eta x}e^{j\xi y}dxdy$$
$$F_+^i(\eta,\xi)=\Psi^i_{++}(\eta,\xi)=\int_0^\infty\int_0^\infty e^{-j\eta_o x-j\xi_o x}e^{j\eta x-j\xi x}dxdy$$
$$F_+^i(\eta,\xi)=-\frac{1}{\eta-\eta_o}\frac{1}{\xi-\xi_o}$$
$$F_-(\eta,\xi)=V_{--}(\eta,\xi)+Y_{-+}(\eta,\xi)+Z_{+-}(\eta,\xi)$$

In the above definitions, the subscript $++$ means that the considered function is (independently on the value of the complex parameter ξ) regular in an upper half-plane of the complex variable η and (independently on the value of the complex parameter η) regular in an upper half-plane of the complex variable ξ.

Similarly:

The subscript $+-$ means that the considered function is (independently on the value of the complex parameter ξ) regular in an upper half-plane of the complex variable η and (independently on the value of the complex parameter η) regular in a lower half-plane of the complex variable ξ.

The subscript $-+$ means that the considered function is (independently on the value of the complex parameter ξ) regular in a lower half-plane of the complex variable η and (independently on the value of the complex parameter η) regular in an upper half-plane of the complex variable ξ.

The subscript $--$ means that the considered function is (independently on the value of the complex parameter ξ) regular in a lower half-plane of the complex variable η and (independently on the value of the complex parameter η) regular in a lower half-plane of the complex variable ξ.

The inverse Fourier transform of a function with subscript $++$, produces a function that in the spatial domain is vanishing as $x<0$ and $y<0$. Similarly it happens for the other subscripts. For instance the subscript $-+$, produces a function that in the spatial domain is vanishing as $x>0$ and $y<0$.

Of course the order of the subscripts is mandatory. For instance the product of the two functions $f_{+-}g_{-+}$ is neither a function $+-$ nor a function $-+$; consequently it is no assured that its inverse Fourier transform is vanishing in the quadrant $x>0, y>0$.

By using the logarithmic decomposition we can write:

$$G(\eta,\xi)=e^{\log[G(\eta,\xi)]}=$$
$$=e^{\{\log[G(\eta,\xi)]\}_{++}+\{\log[G(\eta,\xi)]\}_{+-}+\{\log[G(\eta,\xi)]\}_{-+}+\{\log[G(\eta,\xi)]\}_{--}}= \quad (9)$$
$$=G_{++}(\eta,\xi)G_{+-}(\eta,\xi)G_{-+}(\eta,\xi)G_{--}(\eta,\xi)$$

where the decomposed functions of $\log[G(\eta,\xi)]$ are given by:

$$\{\log[G(\eta,\xi)]\}_{++}=\frac{1}{(2\pi j)^2}\int_{\gamma_1}\int_{\gamma_1}\frac{\log[G(\eta',\xi')]}{(\eta'-\eta)(\xi'-\xi)}d\eta'd\xi'$$
$$\{\log[G(\eta,\xi)]\}_{+-}=-\frac{1}{(2\pi j)^2}\int_{\gamma_1}\int_{\gamma_2}\frac{\log[G(\eta',\xi')]}{(\eta'-\eta)(\xi'-\xi)}d\eta'd\xi'$$
$$\{\log[G(\eta,\xi)]\}_{-+}=-\frac{1}{(2\pi j)^2}\int_{\gamma_2}\int_{\gamma_1}\frac{\log[G(\eta',\xi')]}{(\eta'-\eta)(\xi'-\xi)}d\eta'd\xi'$$
$$\{\log[G(\eta,\xi)]\}_{--}=\frac{1}{(2\pi j)^2}\int_{\gamma_2}\int_{\gamma_2}\frac{\log[G(\eta',\xi')]}{(\eta'-\eta)(\xi'-\xi)}d\eta'd\xi'$$

with γ_1 , γ_2 respectively the "smile" and the "frown" real axis (Fig.2).

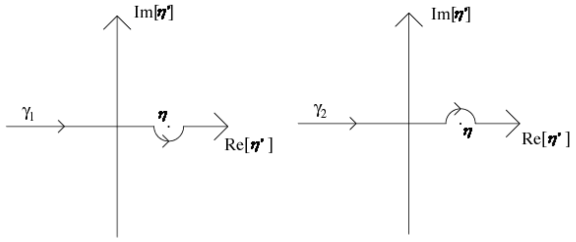

Figure 2: The "smile" (γ_1) and the "frown" (γ_2) real axis in the complex plane η' .

The above integrals have been evaluated explicitly by Radlow [2] (see also [4]). For the above considerations the equation (9) does not constitute a WH factorization. However by using the factors present in the second member of (9), Radlow proposed an explicit factorization $G\,(\eta,\xi)$. Even though several authors gave credit to this solution [3], now it definitively appears that the Radlow factorization is wrong [4].

2 THE FREDHOLM FACTORIZATION

In the natural domain the W-H equation (7) is an integral equations defined by a convolutional kernel. This equation is of first kind. Procedures to reduce it to Fredholm or Volterra equations have been described in literature. However working on the W-H equation in the natural domain seems to be not fruitful. It is better to resort to the functional equations defined in the spectral domain (8).
In particular we can extend to the multidimensional case, the method indicated in [5] to reduce the Wiener-Hopf equation to a Fredholm equation. For this task we introduce the step function $u(x)$ and the projection operators:

$$P \to p(x,y) = u(x)u(y),$$
$$\bar{P} \to 1 - u(x)u(y) =$$
$$\bar{P} \to \bar{p}(x,y) = u(x)u(-y) + u(-x)u(y) + u(-x)u(-y)$$

We observe that:

$$P F_+ = F_+ ,\quad \bar{P} F_+ = (1-P)F_+ = 0 \qquad (10)$$
$$P F_- = 0 \;,\quad \bar{P} F_- = F_- \qquad (11)$$

Multiplying (8) by P and taking into account the first of (10) we get :

$$PGF_+ = PF_+^i \to$$
$$\frac{1}{(2\pi j)^2}\int_{\gamma_1}\int_{\gamma_1}\frac{G(\eta',\xi')}{(\eta'-\eta)(\xi'-\xi)}F_+(\eta',\xi')d\eta' d\xi' = \qquad (12)$$
$$= -\frac{1}{\eta-\eta_o}\frac{1}{\xi-\xi_o}$$

Multiplying the second of (11) by G :

$$\frac{1}{(2\pi j)^2}\int_{\gamma_1}\int_{\gamma_1}\frac{G(\eta,\xi)}{(\eta'-\eta)(\xi'-\xi)}F_+(\eta',\xi')d\eta' d\xi' - G(\eta,\xi)F_+(\eta,\xi) = 0 \qquad (13)$$

Subtracting (13) by (12) we get:

$$G(\eta,\xi)F_+(\eta,\xi) + \qquad (14)$$
$$+\frac{1}{(2\pi j)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{G(\eta',\xi')-G(\eta,\xi)}{(\eta'-\eta)(\xi'-\xi)}F_+(\eta',\xi')d\eta' d\xi' =$$
$$= -\frac{1}{\eta-\eta_o}\frac{1}{\xi-\xi_o}$$

Equation (14) is a two-dimensional Fredholm integral equation of the second kind. Its solution provides the factorization of the two-dimensional kernel $G(\eta,\xi)$.

3 SOLUTION OF THE FREDHOLM EQUATION

As it happens in the one dimensional case, the accuracy of the numerical solutions of (14) considerably increases if one deforms the contour path constituted by the real axis of the $\eta - plane$ ($\xi - plane$) into the straight line $\lambda_\eta\ (\lambda_\xi)$ that joins the points $-jk$ and $+jk$. The introduction of the complex planes w_η, w_ξ (15) yields function-theoretic manipulations of (14), see (16).

$$\eta = -k\cos w_\eta \,,\ \xi = -k\cos w_\xi \qquad (15)$$

$$\hat{G}(u_\eta,u_\xi)\hat{F}_+(u_\eta,u_\xi) +$$
$$+\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{M(u_\eta,u_\xi,u_\eta',u_\xi')\,\mathrm{ch}\,u_\eta'\,\mathrm{ch}\,u_\xi'}{(2\pi)^2(\mathrm{sh}\,u_\eta'-\mathrm{sh}\,u_\eta)(\mathrm{sh}\,u_\xi'-\mathrm{sh}\,u_\xi)}\hat{F}_+(u_\eta',u_\xi')du_\eta' du_\xi' =$$
$$= -\frac{1}{j\,\mathrm{sh}\,u_\eta-\eta_o}\frac{1}{j\,\mathrm{sh}\,u_\xi-\xi_o} \qquad (16)$$

where $\hat{G}(u_\eta,u_\xi) = G(\eta,\xi)$, $\hat{F}_+(u_\eta,u_\xi) = F_+(\eta,\xi)$, $M(u_\eta,u_\xi,u_\eta',u_\xi') = G(\eta',\xi') - G(\eta,\xi)$.
For the one-dimensional factorization, usually (16) is solved by numerical quadrature. For the two-dimensional case the problem of the numerical quadrature of two folded integral remains cumbersome. In order to reduce the number of the

numerical unknowns, the following representation of the function $\hat{F}_+(u_\eta, u_\xi)$ is proposed:

$$\hat{F}_+(u_\eta, u_\xi) = \frac{p_n(u_\eta, u_\xi)}{(j\,\mathrm{sh}\,u_\eta - \eta_o)(j\,\mathrm{sh}\,u_\xi - \xi_o)} \qquad (17)$$

where $p_n(u_\eta, u_\xi)$ is an unknown polynomial of order n whose coefficients can be obtained by applying the collocation method on (16). Using representation (17), we experienced accurate approximate solutions even if we use moderate values of the order n of the polynomial $p_n(u_\eta, u_\xi)$ (the sh at the denominator ensures rapid convergence).

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